B. Math III, Physics IV
Midterm Exam - March, 2006

1. (a) A self-adjoint operator $A$ on a Hilbert space $\mathcal{H}$ is said to be *positive* if $< A\psi,\psi > \geq 0$ for all $\psi \in \mathcal{H}$. If $A$ is a self-adjoint, positive operator, show that, for $\varphi,\psi \in \mathcal{H}$,

$$|< A\varphi,\psi >|^2 \leq < A\varphi,\varphi >< A\psi,\psi >$$

(b) Let $\{ u_n \}_{n=1}^{\infty}$ be an orthonormal sequence in a Hilbert space. Show that the sequence converges weakly to $0$, but it is not convergent in norm.

2. Consider a two dimensional Hilbert space $\mathcal{H}$ with orthonormal basis $\{ e_1, e_2 \}$. Consider the operator $H$ with the following matrix with respect to this basis:

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(a) For an arbitrary real number $\lambda$ determine the matrix for the spectral projection $E_H(\lambda)$.
(b) If $H$ is the Hamiltonian for a system find the probability distribution for the energy in the state

$$\psi = \frac{1}{\sqrt{(a^2+b^2)}} \begin{pmatrix} a \\ b \end{pmatrix}$$

where $a$ and $b$ are real.
(c) If $\psi$ above is the state of the system at time $t = 0$, find the state at a later time $t$.

3. (a) If $H$ is the Hamiltonian of a system on $L^2(R)$ with $H = \frac{P^2}{2m} + V$, show that

$$[X,[H,X]] = \frac{\hbar^2}{m}$$

(b) Let $\psi_n, n = 0,1,2,\ldots$ be the (normalized) eigenfunctions of the Hamiltonian with $H\psi_n = E_n\psi_n$. If the eigenvalues are non-degenerate and if the eigenfunctions span $L^2(R)$, show that

$$\sum_n (E_n - E_0) |< X\psi_n,\psi_0 >|^2 = \frac{\hbar^2}{2m}$$

*Hint:* Compute $< [X,[H,X]]\psi_0,\psi_0 >$ from (a) and use the fact that

$$< u,v > = \sum_n < u,\psi_n >< \psi_n,v >$$

4. Let $\psi_n, n = 0,1,2,....$, be the (normalized) eigenfunctions of the Hamiltonian for the one dimensional harmonic oscillator with $H\psi_n = (n + \frac{1}{2})\hbar\omega\psi_n$. For any complex number $\alpha$ define the state $f_\alpha$ as follows:

$$f_\alpha = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \psi_n$$

(a) Show that $f_\alpha$ is an eigenvector of the lowering operator $A = \sqrt{\frac{1}{2m\hbar\omega}}(m\omega X + iP)$

(b) If $f_\alpha$ is the state of a system at time $t = 0$, show that the state $f_{\alpha,t}$ at any later time $t$ is also an eigenvector of the lowering operator. What is the eigenvalue?

5. Let $L_1, L_2, L_3$ be the components of the angular momentum operator in three dimensions, i.e. $L_1 = X_2P_3 - X_3P_2$, etc. Let $L_\pm = L_1 \pm iL_2$.

(a) If $\psi$ is an eigenvector of $L_3$ with eigenvalue $\lambda$, show that $L_\pm\psi$ (when they are non-zero) are eigenvectors of $L_3$ with corresponding eigenvalues $\lambda \pm \hbar$.

(b) Let $L^2 = (L_1)^2 + (L_2)^2 + (L_3)^2$. Write $L^2$ in terms of $L_+, L_-$ and $L_3$. If $\psi$ is an eigenvector of $L_3$ corresponding to a non-degenerate eigenvalue $\lambda$, then show that $\psi$ is also an eigenvector of $L^2$.

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Some facts about the harmonic oscillator:

$$V(x) = \tfrac{1}{2} m\omega^2 x^2$$

Raising and lowering operators:

$A = \sqrt{\frac{1}{2m\omega\hbar}}(m\omega X + iP)$

$A^* = \sqrt{\frac{1}{2m\omega\hbar}}(m\omega X - iP)$

$A^*\psi_n = (n+1)^{\frac{1}{2}}\psi_{n+1}$

$A\psi_n = n^{\frac{1}{2}}\psi_{n-1}$